PRACTICE EXAM (MID-YEAR) SOLUTIONS

Year 12 Mathematics Methods Exam 1

Part 1

Short-answer questions

Question 1

At $t = 2, v = ?, a = ?$

$$v = \int a\,dt = \int 2\,dt = 2t + c_1$$

At $t = 0, v = 1 \therefore c_1 = 1 \therefore v = 2t + 1$

Similarly,

$$x = \int v\,dt = \int 2t + 1\,dt = t^2 + t + c_2$$

At $t = 0, x = 4 \therefore c_2 = 4 \therefore x = t^2 + t + 4$

At $t = 2$, $v = 5$ cm/s and $x = 10$ cm

[2 marks]

Question 2

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}\left[\sqrt{x^3}\right]_0^1 = \frac{2}{3}$$

[1 mark]

Question 3

a $P(\text{win}) = \dfrac{1}{{}^{40}C_6} = \dfrac{6! \times 34!}{40!}$

[1 mark]

b $P(\text{at least two of the six winning numbers}) = 1 - \dfrac{{}^{34}C_6}{{}^{40}C_6} - \dfrac{{}^{34}C_5 \times {}^{6}C_1}{{}^{40}C_6}$

[1 mark]

Question 4

$$y = \frac{x^3}{3}$$

$$\frac{dy}{dx} = x^2$$

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

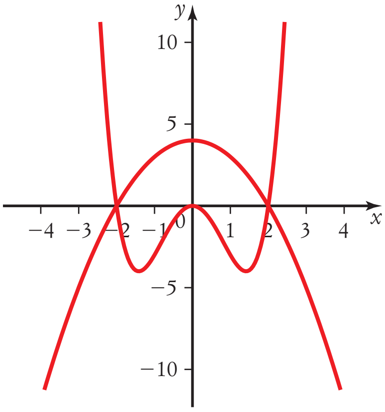
$$\delta y \approx \frac{dy}{dx} \times \delta x$$

At $x = 2$ and $\delta x = 0.01$

$$\delta y \approx 2^2 \times 0.01$$

$$\delta y \approx 0.04$$

Question 5



Points of intersection are $(-2, 0)$ and $(2, 0)$.

Area $= \int_{-2}^{2} (4 - x^2) - (x^2(x^2 - 4))dx$ or

Area $= 2 \times \int_{0}^{2} (4 - x^2) - (x^2(x^2 - 4))dx$

[2 marks]

Question 6

$y = \tan(x)$

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

At $x = 0$, $\frac{dy}{dx} = 1$

$\therefore y = 1 \times x + c$

If $x = 0$, $y = \tan(0) = 0 \therefore c = 0$

Equation of tangent line is $y = x$.

[1 mark]

Question 7

$y = 3\sin\left(x - \frac{\pi}{3}\right)$ for $0 \leq x \leq \pi$.

$$\frac{dy}{dx} = 3\cos\left(x - \frac{\pi}{3}\right)$$

At $\frac{dy}{dx} = 0$, $\left(x - \frac{\pi}{3}\right) = \frac{\pi}{2} \therefore x = \frac{5\pi}{6}$

$\therefore$ Point is $\left(\frac{5\pi}{6}, 3\right)$

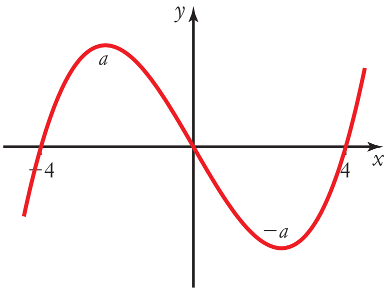
[2 marks]

Question 8

$P(3 \leq x \leq 4) = 0.25$

[1 mark]

Question 9



[2 marks]

Question 10

$f(x) = e^{-3x^2}$ so $f'(x) = -6xe^{-3x^2}$

$e^{-3x^2} > 0$ for $x \in \mathrm{R}$

So $f'(x) < 0$ for $x > 0$

[1 mark]

Part 2

Extended-answer questions

Question 1

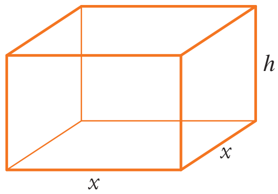
a $V = x^2h = 4$

$h = \dfrac{4}{x^2}$

$\text{SA} = 2x^2 + 4xh$

$\therefore \text{SA} = 2x^2 + 4x \times \dfrac{4}{x^2}$

$\therefore \text{SA} = \dfrac{16}{x} + 2x^2$



b Min SA when $\dfrac{dSA}{dx} = 0$ and $\dfrac{d^2SA}{dx^2} > 0$

$\dfrac{dSA}{dx} = \dfrac{16}{x^2} + 4x$

$\dfrac{d^2SA}{dx^2} = \dfrac{32}{x^3} + 4$

If $\dfrac{dSA}{dx} = 0$, then $\dfrac{16}{x^2} = 4x$

$x^3 = 4$

$x = \sqrt[3]{4}$

Max or min?

$\dfrac{d^2SA}{dx^2} = \dfrac{32}{\sqrt[3]{4}} + 4 > 0 \therefore$ min.

If $x = \sqrt[3]{4}$, $h = ?$

Use $x^2h = 4 \Rightarrow h = \sqrt[3]{4}$

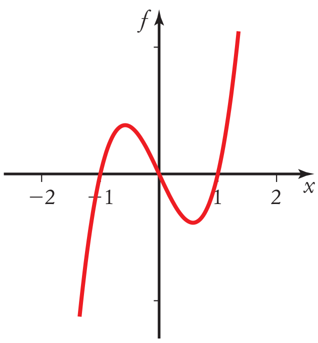
For minimum surface area, $x = \sqrt[3]{4}$, $h = \sqrt[3]{4}$ $(x = h = 1.59)$

[4 marks]

Question 2

$f(x) = x(x - 1)(x + 1) = x^3 - x$

a



b $\int_{-1}^{1}(x^3 - x)dx = \left[\dfrac{x^4}{4} - \dfrac{x^2}{2}\right]_{-1}^{1} = \left(\dfrac{1}{4} - \dfrac{1}{2}\right) - \left(\dfrac{1}{4} - \dfrac{1}{2}\right) = 0$

The areas above and below the axis are equal but as signed areas one is positive and one negative; hence, the sum is zero.

c The area between the function f and the x-axis $= 2 \times \left|\left(\frac{1}{4} - \frac{1}{2}\right)\right| = \frac{1}{2}$ units^2

[4 marks]

Question 3

(1, 1) (1, 2) (1, 3) (1, 4) (2) (3) (4) (5)
(2, 1) (2, 2) (2, 3) (2, 4) (3) (4) (5) (6)
(3, 1) (3, 2) (3, 3) (3, 4) (4) (5) (6) (7)
(4, 1) (4, 2) (4, 3) (4, 4) (5) (6) (7) (8)

Let x be the amount won rolling the two spinners.

Sum x	2	3	4	5	6	7	8
Payout x	\$1	−\$1	−\$1	\$1	−\$1	−\$1	\$1
$P(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

a $E(x) = \frac{1}{16} \times \$1 + \frac{2}{16} \times -\$1 + \frac{3}{16} \times -\$1 + \frac{4}{16} \times \$1 + \frac{3}{16} \times -\$1 + \frac{2}{16} \times -\$1 + \frac{1}{16} \times \$1$

$E(x) = -\$4$

b To make the game fair, \$3 could be paid on the total of 5.

i.e. $E(x) = \frac{1}{16} \times \$1 + \frac{2}{16} \times -\$1 + \frac{3}{16} \times -\$1 + \frac{4}{16} \times \$2 + \frac{3}{16} \times -\$1 + \frac{2}{16} \times -\$1 + \frac{1}{16} \times \$1$

$E(x) = \$0$

Answers will vary.

[5 marks]

Question 4

a $\int \frac{\sqrt{x} - 1}{\sqrt{x}} dx = \int 1 - x^{-\frac{1}{2}} dx$

$= x - 2x^{\frac{1}{2}} + c$

$= x - 2\sqrt{x} + c$

b $\int_1^4 \frac{\sqrt{x} - 1}{\sqrt{x}} dx = \left[x - 2\sqrt{x}\right]_1^4 = (4 - 4) - (1 - 2) = 1$

[4 marks]

Question 5

$$y = \frac{x^2}{\sqrt{x+1}}$$

$$\frac{dy}{dx} = \frac{2x\sqrt{x+1} - \frac{1}{2}(x+1)^{\frac{-1}{2}}x^2}{x+1}$$

$$\frac{dy}{dx} = \frac{x}{x+1}\left(2\sqrt{x+1} - \frac{x}{2\sqrt{x+1}}\right)$$

$$\frac{dy}{dx} = \frac{x}{x+1}\left(\frac{4(x+1)-x}{2\sqrt{x+1}}\right)$$

$$\frac{dy}{dx} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

$$At\ x = 0,\ \frac{dy}{dx} = 0,\ y = 0$$

So the equation of the tangent is $y = 0$

[4 marks]

Question 6

$P(x) = -25x^2 + 300x$

a Max P when $P'(x) = 0$ and $P''(x) < 0$

$P'(x) = -50x + 300$

$P''(x) = -50$ (so max. at any turning point)

If $P'(x) = 0$, then $x = 6$

The local manager should employ 6 checkout workers.

b $P(6) = 900$

The maximum profit is \$900.

[4 marks]